

THE GROTHENDIECK RING OF THE STRUCTURE GROUP OF THE GEOMETRIC FROBENIUS MORPHISM

MARKUS SEVERITT

ABSTRACT. The geometric Frobenius morphism on smooth varieties is an fppf-fiber bundle. We study representations of the structure group scheme. In particular, we describe irreducible representations and compute its Grothendieck ring of finite dimensional representations.

1. INTRODUCTION

Let k be a field of characteristic $p > 0$. Then for all smooth k -varieties X of dimension n , the r -th geometric Frobenius morphism

$$F^r : X \rightarrow X^{(r)}$$

is an fppf-fiber bundle with fibers \mathbb{A}_r^n , the r -th Frobenius kernel of the affine space of dimension n considered as \mathbb{G}_a^n . Now denote

$$R(n, r) := k[\mathbb{A}_r^n] = k[x_1, \dots, x_n] / (x_1^{p^r}, \dots, x_n^{p^r})$$

Let $G(n, r)$ be the automorphism group scheme of $R(n, r)$. Then for each $G(n, r)$ -representation V , there is an associated canonical $X^{(r)}$ -vector bundle by twisting the fiber bundle F^r with V . In order to understand these bundles K-theoretically one needs to understand the Grothendieck ring of $G(n, r)$. The latter one is the purpose of this paper. The topic arose from a correspondence between Pierre Deligne and Markus Rost where Deligne suggested this setting for $n = r = 1$.

We will give a description of the irreducible $G(n, r)$ -representations whose classes form a \mathbb{Z} -basis of $K_0(G(n, r)\text{-rep})$ as an abelian group. Note that $\text{Lie } G(n, r) = \text{Der}_k(R(n, r))$. That is, for $r = 1$, this is the restricted Lie algebra of Cartan type Witt. In fact, the description and the involved computations we give generalize results of [Nak92] for the simple restricted modules of $W_n = \text{Der}_k(R(n, 1))$. In order to get the description of irreducible representations, we need a triangular decomposition $G(n, r) = G^- G^0 G^+$ with $G^0 = \text{GL}_n$. In fact, most of the arguments involved are quite general. Hence we will give them in the abstract notion of *triangulated groups*. Also, this notion turns out to be useful in order to describe the recursive part of our description which passes from $G(n, r)$ to $G(n, r + 1)$. Furthermore, this notion also covers Jantzen's groups $G_r T$ and $G_r B$. Note that the description of irreducible $G(n, 1)$ -representations was already given by Abrams [Abr96], [Abr97]. Unfortunately, one part of our description only works if we exclude the case $\text{char}(k) = 2$.

Our main goal is to describe $K_0(G(n, r)\text{-rep})$ as a surjective image of $r + 1$ copies of $K_0(\text{GL}_n\text{-rep})$ as an abelian group. This involves the de

Rham complex of $R(n, r)$ over k and Cartier's Theorem which computes its cohomology. We will also compute the kernel elements of our surjective map

$$K_0(\mathrm{GL}_n\text{-rep})^{\oplus r+1} \rightarrow K_0(G(n, r)\text{-rep})$$

and introduce a ring structure of the left hand side.

2. BASIC PROPERTIES

Let us first fix some notions we are using. As already said, k is a field of characteristic $p > 0$. By an *algebraic k -group* G we understand a group scheme G , that is a functor from commutative k -algebras to groups, which is representable by a finitely generated Hopf algebra. When we denote $g \in G$ we understand a choice of a commutative k -algebra A and $g \in G(A)$. For the readability, we will suppress A . Furthermore, for $r \geq 0$, we denote by $G^{(r)}$ the r -th *Frobenius twist* and by G_r the kernel of the r -th Frobenius morphism $F^r : G \rightarrow G^{(r)}$.

Now the group scheme

$$G(n, r) = \underline{\mathrm{Aut}}(R(n, r))$$

with

$$R(n, r) := k[\mathbb{A}_r^n] = k[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

is defined to be

$$G(n, r)(A) := \mathrm{Aut}_{A\text{-alg}}(R(n, r)_A)$$

with

$$R(n, r)_A := R(n, r) \otimes_k A = A[x_1, \dots, x_n]/(x_1^{p^r}, \dots, x_n^{p^r})$$

Each element $g \in G(n, r)$ is determined by the images of the x_i . In fact, a choice $g_i \in R(n, r)_A$ for each $i = 1, \dots, n$ defines an element $g \in G(n, r)(A)$ by $g(x_i) = g_i$ if and only if $g_i(0)^{p^r} = 0$ for all i and $J_g \in \mathrm{GL}_n(A)$ where

$$J_g := \left(\frac{\partial g_j}{\partial x_i}(0) \right)_{ij}$$

is the *Jacobian matrix* of g .

As GL_n acts linearly on \mathbb{A}_r^n , we get

$$G^0 := \mathrm{GL}_n \subset G(n, r)$$

as a subgroup. Furthermore, $\mathbb{G}_{a, r}^n$, the r -th Frobenius kernel of \mathbb{G}_a^n , acts on \mathbb{A}_r^n by translation. That is, we get

$$G^- := \mathbb{G}_{a, r}^n \subset G(n, r)$$

as a subgroup. Now denote

$$\begin{aligned} G^+ &:= \{g \in G(n, r) \mid \forall i : g(x_i)(0) = 0, J_g = \mathrm{id}\} \\ &= \{g \in G(n, r) \mid \forall i : g(x_i) = x_i + \sum_{I, \deg(I) \geq 2} \lambda_I x^I\} \end{aligned}$$

where $I \in \{0, \dots, p^r - 1\}^n$ is a multi-index with the usual degree $\deg(I)$ and $x^I \in R(n, r)$ is the corresponding monomial. Note that $G^+ \cong \mathbb{A}^N$ for an $N \in \mathbb{N}$ as a scheme. These three subgroups provide a triangular decomposition

$$G(n, r) = G^- G^0 G^+$$

That is, the multiplication $m : G^- \times G^0 \times G^+ \rightarrow G$ is an isomorphism of k -schemes.

Moreover, for all $1 \leq i \leq n$, denote

$$U_i = U_i(n, r) := \{g \in G(n, r) \mid \forall j : g(x_j)(0)^{p^i} = 0\} \subset G(n, r)$$

These subschemes are in fact subgroups who afford the triangular decomposition

$$U_i = G_i^- G^0 G^+$$

Note that $G_i^- = \mathbb{G}_{a,i}^n$.

As we already noticed,

$$\mathrm{Lie}(G(n, r)) = \mathrm{Der}_k(R(n, r))$$

the self-derivations of $R(n, r)$ by [DG80, II§4,2.3 Proposition]. A canonical basis of this Lie algebra is given by the operators

$$\delta_{(i, x^I)} := x^I \frac{\partial}{\partial x_i}, \quad I \in \{0, \dots, p^r - 1\}^n$$

3. TRIANGULATED GROUPS

Now we will introduce the notion of triangulated groups and triangulated morphisms. It turned out to be a convenient notion in order to study the group schemes $G(n, r)$. Triangular decompositions are a standard tool in algebraic Lie theory. This notion is meant to catch some of their properties in an abstract way.

Definition 3.1. Let H be an algebraic k -group. A *pretriangulation* of H is a collection of three subgroups (H^-, H^0, H^+) such that the multiplication map

$$m : H^- \times H^0 \times H^+ \rightarrow H$$

is an isomorphism of k -schemes. We shortly denote this by

$$H = H^- H^0 H^+$$

As an example, we have

$$G(n, r) = G^- G^0 G^+$$

as well as $U_i(n, r) = G_i^- G^0 G^+$.

Furthermore for each split reductive group G , we get for the r -th Frobenius kernel

$$G_r = U_r^- T_r U_r^+$$

where $T \subset G$ is a maximal torus and $U^\pm \subset G$ the unipotent subgroups.

Definition 3.2. Let $G = G^- G^0 G^+$ and $H = H^- H^0 H^+$ be pretriangulated and $f : G \rightarrow H$ a group homomorphism. Then f is said to be *triangulated*, if for all $\alpha \in \{-, 0, +\}$ the restriction of f to G^α factors through H^α .

We denote this factorization by $f^\alpha : G^\alpha \rightarrow H^\alpha$ and we write

$$f = f^- f^0 f^+$$

As an example consider a pretriangulation $H = H^- H^0 H^+$. Then the r -th Frobenius twist of H is pretriangulated by $H^{(r)} = (H^-)^{(r)} (H^0)^{(r)} (H^+)^{(r)}$ and the r -th Frobenius morphisms

$$F_H^r : H \rightarrow H^{(r)}$$

is triangulated with $(F_H^r)^\alpha = F_{H^\alpha}^r$.

The following Lemma is straightforward.

Lemma 3.3. *Let $f : G \rightarrow H$ be triangulated. Then the following holds:*

- (1) *The kernel of f is pretriangulated by*

$$\text{Ker}(f) = \text{Ker}(f^-) \text{Ker}(f^0) \text{Ker}(f^+)$$

- (2) *The image of f is pretriangulated by*

$$\text{Im}(f) = \text{Im}(f^-) \text{Im}(f^0) \text{Im}(f^+)$$

- (3) *The closed immersion $\text{Im}(f) \hookrightarrow H$ is triangulated.*

Note that under the isomorphism $G / \text{Ker}(F) \cong \text{Im}(f)$, we also obtain a pretriangulation

$$G / \text{Ker}(f) = G^- / \text{Ker}(f^-) G^0 / \text{Ker}(f^0) G^+ / \text{Ker}(f^+)$$

As an example we take a pretriangulation $H = H^- H^0 H^+$ and the r -th Frobenius morphism $F_H^r : H \rightarrow H^{(r)}$. Then we get that the r -th Frobenius kernel of H is pretriangulated by

$$H_r = H_r^- H_r^0 H_r^+$$

Our next aim is to describe irreducible representation of a pretriangulated group H in terms of H^0 . In order to do that, we need the following.

Definition 3.4. A pretriangulation $H = H^- H^0 H^+$ is called a *triangulation* if the following statements hold:

- (1) The following products are semi direct by conjugation

$$B^- := H^- \rtimes H^0 \text{ and } B^+ := H^0 \ltimes H^+$$

- (2) H^- and H^+ are unipotent.

- (3) H^- is finite.

Note that this definition is not symmetric. As an example, the pretriangulation $G(n, r) = G^- G^0 G^+$ is also a triangulation.

For a triangulation $H = H^- H^0 H^+$, we denote the group homomorphisms

- (1) the projections $\pi^\pm : B^\pm \rightarrow H^0$
(2) the inclusions $j^\pm : B^\pm \hookrightarrow H$

and the functor

$$\mathcal{I} := (j^+)_* (\pi^+)^* : H^0\text{-rep} \rightarrow H\text{-rep}$$

between categories of finite dimensional representations. Here

$$(\pi^+)^* : H^0\text{-rep} \rightarrow B^+\text{-rep}$$

and

$$(j^+)_* = \text{Mor}_{B^+}(H, -) : B^+\text{-rep} \rightarrow H\text{-rep}$$

is induction. That is

$$(j^+)_*(V) = \{f \in \text{Mor}(H, V) \mid f(hg) = hf(g) \ \forall g \in H, \ \forall h \in B^+\}$$

and the H -action is induced by right translation on H . The functor $(j^+)_*$ preserves finiteness as

$$\mathrm{Mor}_{B^+}(H, -) \cong \mathrm{Mor}(H^-, -)$$

by using the decomposition $H = B^+H^-$ and the finiteness of H^- . In order to express the H -action, let us denote for $h \in H$ the decomposition

$$h = h_+ h_0 h_-$$

according to the decomposition $H = B^+H^- = H^+H^0H^-$. The following Lemma is straightforward.

Lemma 3.5. *Let $H = H^-H^0H^+$ be a triangulation and V an H^0 -representation.*

(1) *Under the isomorphism*

$$\mathcal{I}(V) \cong \mathrm{Mor}(H^-, V)$$

the H -action translates as follows: For all $h \in H$, $a \in H^-$, and $f : H^- \rightarrow V$

$$(hf)(a) = (ah)_0 f((ah)_-)$$

(2) *If we restrict to B^- , we get*

$$(j^-)^* \mathcal{I}(V) \cong k[H^-] \otimes_k V$$

Here H^- acts on $k[H^-]$ by the right regular representation and trivial on V and H^0 acts on $k[H^-]$ by conjugation and as given on V .

In particular, we get

$$\mathcal{I}(V)^{H^-} \cong (k[H^-] \otimes_k V)^{H^-} = k[H^-]^{H^-} \otimes_k V \cong V$$

as H^0 -representations.

Example 3.6. Let us consider the group scheme $G(n, r)$ and the triangulation $G(n, r) = G^-G^0G^+$. Recall that $G^- = (\mathbb{G}_{a,r})^n$ and $G^0 = \mathrm{GL}_n$. Then for all $g \in G(n, r)$, we get $g_- = g(0) \in (\mathbb{G}_{a,r})^n$ and $g_0 = J_g \in \mathrm{GL}_n$. As $R(n, r) = k[(\mathbb{G}_{a,r})^n]$, we get for each GL_n -representation V that

$$\mathcal{I}(V) \cong R(n, r) \otimes_k V$$

as k -vector spaces. The $G(n, r)$ -action reads as follows:

$$g(f \otimes v) = \left(\frac{\partial g(x_j)}{\partial x_i} \right)_{ij} (g(f) \otimes v)$$

for all $g \in G(n, r)$, $f \in R(n, r)$, and $v \in V$. Note that this uses the fact that for $g \in G(n, r)(A)$ we get $\left(\frac{\partial g(x_j)}{\partial x_i} \right)_{ij} \in \mathrm{GL}_n(R(n, r)_A)$ which acts on $R(n, r)_A \otimes_k V$.

In particular, take $V = k$ with the trivial GL_n -action. Then we get $\mathcal{I}(k) \cong R(n, r)$ together with the standard action of $G(n, r)$ on $R(n, r)$. That is, the standard action can be recovered from the triangulation.

Now the functor \mathcal{I} gives rise to the following description of irreducible H -representations.

Proposition 3.7. *Let $H = H^- H^0 H^+$ be a triangulation. Then the maps*

$$\text{soc } \mathcal{I} : \{\text{irred. } H^0\text{-rep}\} / \cong \longrightarrow \{\text{irred. } H\text{-rep}\} / \cong$$

and

$$(-)^{H^-} : \{\text{irred. } H\text{-rep}\} / \cong \longrightarrow \{\text{irred. } H^0\text{-rep}\} / \cong$$

are well-defined and inverse to each other.

Proof. Let V be an irreducible H^0 -representation. As H^- is unipotent, a standard argument by taking H^- -invariants and using $\mathcal{I}(V)^{H^-} \cong V$ shows that $\text{soc } \mathcal{I}(V)$ is an irreducible H -representation. Also, $\text{soc } \mathcal{I}(V)^{H^-} \cong V$ follows. Now let W be an irreducible H -representation. As H^+ is unipotent $(W^\vee)^{H^+} \neq 0$. Thus there is an irreducible H^0 -representation $V \neq 0$ such that $V^\vee \subset (W^\vee)^{H^+}$. Hence by dualization

$$0 \neq \text{Hom}_{B^+}((j^+)^* W, (\pi^+)^* V) \cong \text{Hom}_H(W, \mathcal{I} V)$$

This shows $W \cong \text{soc } \mathcal{I}(V)$ and finishes the proof. \square

Finally, we extend the notion of triangulations as follows.

Definition 3.8. A triangulation $H = H^- H^0 H^+$ is called an r -triangulation if H^- is of height $\leq r$. That is, H^- equals its r -th Frobenius kernel:

$$H^- = (H^-)_r$$

Note that an r -triangulation is also an $r+1$ -triangulation. As an example, the triangulation $G(n, r) = G^- G^0 G^+$ is also an r -triangulation as $G^- = \mathbb{G}_{a,r}^n$. Moreover $U_i(n, r) = G_i^- G^0 G^+$ is an i -triangulation.

Now for an r -triangulation $H = H^- H^0 H^+$, the r -th Frobenius factors as

$$F_H^r : H \rightarrow (B^+)^{(r)} \subset H^{(r)}$$

which provides the group homomorphism

$$P_r := \pi^+ \circ F_H^r : H \rightarrow (H^0)^{(r)}$$

In the example $G(n, r)$, this computes as

$$P_r(g) = F_{\text{GL}_n}^r(J_g) = \left(\left(\frac{\partial g(x_j)}{\partial x_i}(0) \right)^{p^r} \right)_{ij}$$

If we consider $(H^0)^{(r)}$ triangulated with trivial \pm -factors, P_r is triangulated with P_r^\pm trivial and $P_r^0 = F_{H^0}^r$. Thus

$$\text{Ker}(P_r) = H^- H_r^0 H^+$$

Moreover, P_r has the following very useful property.

Lemma 3.9. *Let $H = H^- H^0 H^+$ be an r -triangulation such that H^0 is reduced. Then P_r induces an isomorphism*

$$H / \text{Ker}(P_r) \cong (H^0)^{(r)}$$

Proof. This follows from the triangulated structure of P_r , Lemma 3.3 and the fact that $F_{H^0}^r$ induces an isomorphism

$$H^0 / (H^0)_r \cong (H^0)^{(r)}$$

if H^0 is reduced [Jan03, I.9.5]. \square

As an immediate consequence, we get that for H^0 reduced, the functor

$$P_r^* : (H^0)^{(r)}\text{-rep} \longrightarrow H\text{-rep}$$

preserves irreducible representations.

Notation 3.10. For any algebraic k -group G and a $G^{(r)}$ -representation W , we denote the r -th Frobenius twist by

$$W^{[r]} := (F_G^r)^* W$$

Then we get the following computational rule.

Lemma 3.11. *Let $H = H^- H^0 H^+$ be an r -triangulation, V an H^0 -representation, and W an $(H^0)^{(r)}$ -representation. Then*

$$\mathcal{I}(V \otimes_k W^{[r]}) \cong \mathcal{I}(V) \otimes_k P_r^* W$$

as H -representations.

Proof. The claim follows from the Tensor Identity [Jan03, I.3.6] for induction and the fact that $P_r^* W|_{H^0} = W^{[r]}$. \square

This provides the following Proposition which reads as and uses Steinberg's Tensor Product Theorem [Jan03, II.3.16, II.3.17]. For this, we need the following notations where for split reductive groups we follow [Jan03].

Notation 3.12. Let $H = H^- H^0 H^+$ be a triangulation such that H^0 is split reductive. Denote by $T \subset H^0$ a split maximal torus and by $X(T)_+$ the dominant weights. Furthermore denote by S the simple roots and for $r \geq 1$

$$X_r(T) := \{\lambda \in X(T) \mid \forall \alpha \in S : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r\}$$

For $\lambda \in X(T)_+$, we denote the associated irreducible H^0 -representation by $L(\lambda)$. Then, according to Proposition 3.7, we denote the associated irreducible H -representation by

$$L(\lambda, H) := \text{soc } \mathcal{I}(L(\lambda))$$

Note that for an r -triangulation $H = H^- H^0 H^+$ with H^0 split reductive, we get for all $\lambda \in X(T)_+$

$$L(p^r \lambda, H) \cong P_r^* L(\lambda)$$

as P_r^* preserves irreducible representations and

$$(P_r^* L(\lambda))^{H^-} = P_r^* L(\lambda)|_{H^0} = L(\lambda)^{[r]} \cong L(p^r \lambda)$$

Proposition 3.13. *Let $H = H^- H^0 H^+$ be an r -triangulation such that H^0 is split reductive, $\lambda \in X_r(T)$ and $\mu \in X(T)_+$. Then*

$$L(\lambda + p^r \mu, H) \cong L(\lambda, H) \otimes_k P_r^* L(\mu) \cong L(\lambda, H) \otimes_k L(p^r \mu, H)$$

Proof. By Steinberg's Tensor Product Theorem

$$L(\lambda + p^r \mu) \cong L(\lambda) \otimes_k L(\mu)^{[r]}$$

Then the previous Lemma provides

$$\mathcal{I}(L(\lambda + p^r \mu)) \cong \mathcal{I}(L(\lambda)) \otimes_k P_r^* L(\mu)$$

As $P_r^* L(\mu) = (P_r^* L(\mu))^{H^-}$, the result follows by taking socles. \square

Note that examples of such r -triangulations with reductive H^0 are given by Jantzen's groups $G_r T$ and $G_r B$. The Proposition just generalizes results which are already known for these groups.

4. TRANSFER HOMOMORPHISMS

In order to prepare the description of the irreducible $G(n, r)$ -representations, we will introduce several *transfer homomorphisms*. They will be between the $G(n, r)$, $U_i(n, r)$, and $G(n, r)^0 \cong \mathrm{GL}_n$ and their Frobenius twists respectively.

In the previous section, we already introduced the morphism

$$P_r : G(n, r) \rightarrow (\mathrm{GL}_n)^{(r)}$$

arising from the r -triangulation $G(n, r) = G^- G^0 G^+$. We also saw that for all $\lambda \in X(T)_+$ we have

$$P_r^* L(\lambda) \cong L(p^r \lambda, G(n, r))$$

Now, for $r \geq 2$ we also introduce a transfer morphism

$$T_r : G(n, r) \rightarrow G(n, r-1)^{(1)}$$

as follows: Let $S \subset R(n, r)$ be the subalgebra generated by $x_1^{p^{r-1}}, \dots, x_n^{p^{r-1}}$. Now let $g \in G(n, r) = \underline{\mathrm{Aut}}(R(n, r))$. Then S is invariant under g . Thus we get an induced automorphism on

$$R(n, r) \otimes_{S, (-)^p} k \cong R(n, r-1) \otimes_{k, (-)^p} k = R(n, r-1)^{(1)}$$

which defines $T_r(g)$. Note that if $F_a : R(n, r) \rightarrow R(n, r)$ denotes the *arithmetic Frobenius*, then

$$T_r(g)(x_i \otimes 1) = g(x_i) \otimes 1 = (F_a(g(x_i)))(x_i \otimes 1)$$

Now T_r is triangulated as $(T_r)^-$ is just the first Frobenius morphism

$$F^1 : \mathbb{G}_{a,r}^n \rightarrow (\mathbb{G}_{a,r-1}^n)^{(1)}$$

and $(T_r)^0$ is also the first Frobenius morphism

$$F^1 : \mathrm{GL}_n \rightarrow (\mathrm{GL}_n)^{(1)}$$

Note that $(T_r)^+$ is not just the first Frobenius as we set $x_i^{p^{r-1}} = 0$ for all $i = 1, \dots, n$.

Lemma 4.1. *For all $r \geq 2$, the morphism $T_r : G(n, r) \rightarrow G(n, r-1)^{(1)}$ induces an isomorphism*

$$G(n, r) / \mathrm{Ker}(T_r) \xrightarrow{\cong} G(n, r-1)^{(1)}$$

Proof. By the triangulated structure of T_r and Lemma 3.3, it suffices to show the claim for $(T_r)^-, (T_r)^0, (T_r)^+$ separately. For $(T_r)^-$ it follows by [Jan03, I.9.5] which states that it induces an isomorphism

$$\mathbb{G}_{a,r}^n / \mathbb{G}_{a,1}^n \cong (\mathbb{G}_{a,r-1}^n)^{(1)}$$

Also by [Jan03, I.9.5], it follows for $(T_r)^0$: It induces an isomorphism

$$\mathrm{GL}_n / (\mathrm{GL}_n)_1 \cong (\mathrm{GL}_n)^{(1)}$$

It is left to show that the closed immersion

$$T_r^+ : G(n, r)^+ / \text{Ker}(T_r^+) \hookrightarrow (G(n, r-1)^+)^{(1)}$$

is an isomorphism. The describing ideal of this immersion is the kernel of the morphism

$$(T_r^+)^{\#} : k[G(n, r-1)^+]^{(1)} \rightarrow k[G(n, r)^+]$$

As the parameters for $G(n, r)^+$ are free and T_r acts as the p -th power on the parameters, $(T_r^+)^{\#}$ is injective which shows the claim. \square

As an immediate consequence, we get that

$$T_r^* : G(n, r-1)^{(1)}\text{-rep} \longrightarrow G(n, r)\text{-rep}$$

preserves irreducible representations. More concretely

$$T_r^* L(\lambda, G(n, r-1)^{(1)}) \cong L(p\lambda, G(n, r))$$

for all $\lambda \in X(T)_+$. This follows by Proposition 3.7 and the observation that

$$\begin{aligned} L(p\lambda) = L(\lambda)^{[1]} &= (F_{\text{GL}_n}^1)^* \left(L(\lambda, G(n, r-1)^{(1)})^{(G(n, r-1)^{(1)})^-} \right) \\ &\subset (T_r^* L(\lambda, G(n, r-1)^{(1)}))^{G(n, r)^-} \end{aligned}$$

Furthermore we introduce the transfer homomorphisms

$$t_{r,i} : U_i(n, r) \rightarrow G(n, i)$$

for all $1 \leq i \leq r$. Recall that

$$U_i = \{f \in G(n, r) \mid f(0) \in \mathbb{G}_{a,i}^n\}$$

So let $g \in U_i$. Then it induces an isomorphism on $R(n, i)$ which we denote by $t_{r,i}(g)$. Note that

$$t_{r,i}(g)(x_i) \equiv g(x_i) \pmod{(x_1^{p^i}, \dots, x_n^{p^i})}$$

Also $t_{r,i}$ is triangulated: The restriction to G_i^- and G^0 is just the identity.

Finally, we discuss how the maps P_r , T_r , $t_{r,i}$ are related. First note that for all $r \geq 2$, the diagram

$$\begin{array}{ccc} G(n, r) & \xrightarrow{T_r} & G(n, r-1)^{(1)} \\ & \searrow P_r & \downarrow P_{r-1} \\ & & (G^0)^{(r)} \end{array}$$

commutes. Furthermore, for all $1 \leq i \leq r$, the diagram

$$\begin{array}{ccc} U_i(n, r) & \xrightarrow{t_{r,i}} & G(n, i) \\ & \searrow P_i & \downarrow P_i \\ & & (G^0)^{(i)} \end{array}$$

commutes.

We again denote $G^- = G(n, r)^-$ and $U_i = U_i(n, r) \subset G(n, r)$. Recall that $U_i^- = G_i^-$. Our next aim is to study the induction functor $\text{ind}_{U_i}^{G(n, r)}$ and its relation to the induced functors of the three morphism types.

Lemma 4.2. *For all $1 \leq i \leq r$, we get for the induction functor*

$$\text{res}_{G^-}^{G(n,r)} \circ \text{ind}_{U_i}^{G(n,r)} = \text{ind}_{G_i^-}^{G^-} \circ \text{res}_{G_i^-}^{U_i}$$

Furthermore $\text{ind}_{U_i}^{G(n,r)}$ is exact.

Proof. We use the morphism description of the induction $\text{ind}_{U_i}^{G(n,r)}$. Then we obtain for all U_i -representations V that

$$\begin{aligned} \text{ind}_{U_i}^{G(n,r)} V &= \{f \in \text{Mor}(G(n,r), V) \mid f(ug) = uf(g) \ \forall u \in U_i\} \\ &= \{f \in \text{Mor}(G^-, V) \mid f(cg) = cf(g) \ \forall c \in G_i^-\} \end{aligned}$$

by using the decomposition $G(n,r) = B^+ \times G^-$ and $U_i = B^+ \times G_i^-$ where $B^+ = G(n,r)^0 \ltimes G(n,r)^+$. The restriction of this to G^- coincides with

$$\text{ind}_{G_i^-}^{G^-} \text{res}_{G_i^-}^{U_i} V$$

as the G^- -action is given by right translation. This shows the first claim. Now $\text{ind}_{G_i^-}^{G^-}$ is exact by [Jan03, I.5.13, I.9.5]. This implies the exactness of $\text{ind}_{U_i}^{G(n,r)}$. \square

We start with the relation of $\text{ind}_{U_i}^{G(n,r)}$ to the \mathcal{I} -functors. Recall that $G(n,r)$ is r -triangulated and U_i is i -triangulated. So let us denote the \mathcal{I} -functor for a j -triangulated group as \mathcal{I}_j .

Lemma 4.3. *For all $1 \leq i \leq r$, both triangles of the diagram*

$$\begin{array}{ccccc} & & G(n,i)\text{-rep} & & \\ & \nearrow \mathcal{I}_i & & \searrow t_{r,i}^* & \\ G^0\text{-rep} & \xrightarrow{\mathcal{I}_i} & U_i(n,r)\text{-rep} & & \\ & \searrow \mathcal{I}_r & & \swarrow \text{ind}_{U_i}^{G(n,r)} & \\ & & G(n,r)\text{-rep} & & \end{array}$$

commute.

Proof. The commutativity of the upper triangle follows immediately from $G(n,i)^- = U_i(n,r)^-$ and Lemma 3.5.

The commutativity of the lower triangle follows from $G(n,r) = B^+ \times G^-$, $U_i = B^+ \times G_i^-$, and the transitivity of induction [Jan03, I.3.5]. \square

Finally there is a more complicated relation of the induction $\text{ind}_{U_i}^{G(n,r)}$ to the functors P_i^* , T_j^* , and \mathcal{I} :

Lemma 4.4. *For all $1 \leq i \leq r$, the triangle and the square of the following diagram commute up to functor isomorphism*

$$\begin{array}{ccccc}
 & & G(n, i)\text{-rep} & & \\
 & \nearrow^{P_i^*} & & \nwarrow_{t_{r,i}^*} & \\
 (G^0)^{(i)}\text{-rep} & \xrightarrow{P_i^*} & U_i(n, r)\text{-rep} & & \\
 \mathcal{I}_{r-i} \downarrow & & \downarrow \text{ind}_{U_i}^{G(n,r)} & & \\
 G(n, r-i)^{(i)}\text{-rep} & \xrightarrow{(T^i)^*} & G(n, r)\text{-rep} & &
 \end{array}$$

Here $T^i : G(n, r) \rightarrow G(n, r-i)^{(i)}$ is the composition

$$T^i := T_{r-(i-1)}^{(i-1)} \circ \cdots \circ T_r$$

Proof. The commutativity of the triangle follows from $P_i \circ t_{r,i} = P_i$.

For the commutativity of the square, note that the morphism

$$T^i = T_{r-(i-1)}^{(i-1)} \circ \cdots \circ T_r$$

is triangulated with

$$(T^i)^- = F_{G^-}^i : G^- \rightarrow (G^-)_{r-i}^{(i)}$$

and

$$(T^i)^0 = F_{G^0}^i : G^0 \rightarrow (G^0)^{(i)}$$

Recall the morphism description of the functor \mathcal{I} from Lemma 3.5. Let V be a $(G^0)^{(i)}$ -representation. On one hand

$$(T^i)^* \mathcal{I}_{r-i}(V) = \text{Mor}((G^-)_{r-i}^{(i)}, V)$$

as $(G(n, r-i)^{(i)})^- = (G^-)_{r-i}^{(i)}$. On the other hand, G_i^- operates trivially on $P_i^* V$ which implies

$$\begin{aligned}
 \text{ind}_{U_i}^{G(n,r)} P_i^* V &= \{f \in \text{Mor}(G^-, P_i^* V) \mid f(cg) = cf(g) \ \forall c \in G_i^-\} \\
 &= \text{Mor}(G^-/G_i^-, P_i^* V)
 \end{aligned}$$

(cf. the proof of Lemma 4.2). As the i -th Frobenius $F_{G^-}^i : G^- \rightarrow (G^-)_{r-i}^{(i)}$ induces an isomorphism $G^-/G_i^- \cong (G^-)_{r-i}^{(i)}$, it induces a natural linear isomorphism

$$(T^i)^* \mathcal{I}_{r-i}(V) = \text{Mor}((G^-)_{r-i}^{(i)}, V) \xrightarrow{(F^i)^*} \text{Mor}_{G_i^-}(G^-, P_i^* V) = \text{ind}_{U_i}^{G(n,r)} P_i^* V$$

This isomorphism is in fact $G(n, r)$ -equivariant which can be seen by using the triangulated structure of T^i . \square

5. DIFFERENTIALS AND CARTIER'S THEOREM

We are now going to introduce some concrete $G(n, r)$ -representations which play a major role in the description of the irreducible representations. We consider the Kähler-differentials

$$\Omega_r := \Omega_{R(n,r),k} = \bigoplus_{i=1}^n R(n, r) dx_i$$

We claim that this is a canonical $G(n, r)$ -representation.

Notation 5.1. For any $g \in G(n, r)$ and any $R(n, r)$ -module M , we denote by $M^{(g)}$ the module *twisted by g* , that is

$$x *^{(g)} m := g(x)m$$

We obtain that

$$R(n, r) \xrightarrow{g} R(n, r)^{(g)} \xrightarrow{d} \Omega_r^{(g)}$$

is an $R(n, r)$ -derivation which induces an $R(n, r)$ -module automorphism

$$\partial g : \Omega_r \rightarrow \Omega_r^{(g)}$$

This reads as

$$\partial g(f dx_i) = g(f) dg(x_i)$$

and provides a canonical $G(n, r)$ -action on Ω_r as a k -vector space. Moreover this operation extends to exterior and symmetric powers over $R(n, r)$ as well as tensor products. In particular, we obtain a representation by the i -th higher differentials

$$\Omega_r^i := \Lambda_{R(n, r)}^i \Omega_r = \bigoplus_{j_1 < \dots < j_i} R(n, r) dx_{j_1} \wedge \dots \wedge dx_{j_i}$$

They are connected by the *de Rham complex*

$$0 \rightarrow R(n, r) \xrightarrow{d_1} \Omega_r^1 \xrightarrow{d_2} \dots \xrightarrow{d_n} \Omega_r^n \rightarrow 0$$

The differential maps are defined by

$$d_i(f dx_{j_1} \wedge \dots \wedge dx_{j_i}) := df \wedge dx_{j_1} \wedge \dots \wedge dx_{j_i}$$

In fact the maps d_i are $G(n, r)$ -equivariant which can be shown by induction on i .

Remark 5.2. Note that with $U = k^n$, we canonically get

$$\Omega_r^i \cong R(n, r) \otimes_k \Lambda^i U \cong \mathcal{I}_r(\Lambda^i U)$$

according to Example 3.6. Hence, for all $(\mathrm{GL}_n)^{(r)}$ -representations V , we obtain

$$\mathcal{I}_r(\Lambda^i U \otimes V^{[r]}) \cong \Omega_r^i \otimes P_r^* V$$

by Lemma 3.11. That is, we also have a twisted de Rham complex $\Omega_r^\bullet \otimes P_r^* V$.

Now we use the transfer morphism $t_{r, j} : U_j(n, r) \rightarrow G(n, j)$ in order to compare the de Rham complexes Ω_j^\bullet and Ω_r^\bullet by the functor

$$\mathrm{ind}_{U_j}^{G(n, r)} \circ t_{r, j}^* : G(n, j)\text{-rep} \longrightarrow G(n, r)\text{-rep}$$

Proposition 5.3. *For all $1 \leq j < r$, we get*

$$\mathrm{ind}_{U_j}^{G(n, r)}(t_{r, j}^* \Omega_j^\bullet) \cong \Omega_r^\bullet$$

as complexes. Furthermore

$$\mathrm{ind}_{U_j}^{G(n, r)}(t_{r, j}^* H^i(\Omega_j^\bullet)) \cong H^i(\Omega_r^\bullet)$$

for all $0 \leq i \leq n$.

Proof. By Lemma 4.3 and the previous remark, we get canonical isomorphisms

$$\begin{aligned} \mathrm{ind}_{U_j}^{G(n, r)}(t_{r, j}^* \Omega_j^i) &\cong \mathrm{ind}_{U_j}^{G(n, r)}(t_{r, j}^* \mathcal{I}_j \wedge^i U) \\ &\cong \mathcal{I}_r(\wedge^i U) \\ &\cong \Omega_r^i \end{aligned}$$

Similarly, one can check by a tedious exercise that also

$$\mathrm{ind}_{U_j}^{G(n, r)}(t_{r, j}^*(d_i : \Omega_j^{i-1} \rightarrow \Omega_j^i)) \cong (d_i : \Omega_r^{i-1} \rightarrow \Omega_r^i)$$

The claim about the cohomology follows from the exactness of $\mathrm{ind}_{U_j}^{G(n, r)}$ Lemma 4.2. \square

That is, we can compute the cohomology of the complex Ω_r^\bullet by the cohomology of the complex Ω_1^\bullet .

Before we do this, we need an additional observation. Let $f^r : R(n, r) \rightarrow R(n, r)$ the r -th power of the absolute Frobenius. It factors as

$$R(n, r) \xrightarrow{f^r} k \hookrightarrow R(n, r)$$

as $P^{p^r} = P(0)^{p^r}$ for all $P \in R(n, r)$. This provides induced $G(n, r)$ -representations

$$\Omega_r^i \otimes_{R(n, r), f^r} k \cong \wedge^i U \otimes_{k, f^r} k = \wedge^i U^{(r)}$$

where again $U = k^n$.

Lemma 5.4. *For all $1 \leq i \leq n$, we get*

$$\Omega_r^i \otimes_{R(n, r), f^r} k \cong P_r^* \wedge^i U^{(r)}$$

Proof. Observe that the group homomorphism

$$G(n, r) \rightarrow \mathrm{GL}(U^{(r)}) = (\mathrm{GL}_n)^{(r)}$$

corresponding to the $G(n, r)$ -representation $\Omega_r \otimes_{R(n, r), f^r} k$ coincides with P_r . This provides the claim for $i = 1$. The claim for $i \geq 2$ follows from this by compatibility with exterior powers. \square

We get a representation-theoretic reformulation of Cartier's famous theorem about the cohomology of the de Rham complex. It follows from its proof in [Kat70, Theorem 7.2] and the previous lemma.

Theorem 5.5 (Cartier). *There is a unique collection of isomorphisms of $G(n, 1)$ -representations*

$$C^{-1} : P_1^* \wedge^i U^{(1)} \rightarrow H^i(\Omega_1^\bullet)$$

which satisfies

- (1) $C^{-1}(1) = 1$
- (2) $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$
- (3) $C^{-1}(df \otimes 1) = [f^{p-1} df] \in H^1(\Omega_1^\bullet)$

Remark 5.6. In fact, the proof in [Kat70] does not provide the property that the C^{-1} are $G(n, 1)$ -equivariant. But by property (2), it suffices to check it for $i = 1$ which follows from property (3).

As announced before, we can deduce a computation of the cohomology of Ω_r^\bullet for $r \geq 2$ with help of the transfer morphism $T_r : G(n, r) \rightarrow G(n, r-1)^{(1)}$.

Corollary 5.7. *For all $r \geq 2$ and $1 \leq i \leq n$, we get an isomorphism*

$$H^i(\Omega_r^\bullet) \cong T_r^*((\Omega_{r-1}^i)^{(1)})$$

of $G(n, r)$ -representations.

Proof. According to Proposition 5.3, Cartier's Theorem, and Lemma 4.4, we obtain

$$\begin{aligned} H^i(\Omega_r^\bullet) &\cong \operatorname{ind}_{U_1}^{G(n, r)}(t_{r,1}^*(H^i(\Omega_1^\bullet))) \\ &\cong \operatorname{ind}_{U_1}^{G(n, r)}(t_{r,1}^*(P_1^* \Lambda^i U^{(1)})) \\ &\cong T_r^*(\mathcal{I}_{r-1}(\Lambda^i U^{(1)})) \\ &\cong T_r^*((\Omega_{r-1}^i)^{(1)}) \end{aligned}$$

Whence the claim. \square

Finally we want to twist the de Rham complex Ω_r^\bullet with an $(\mathrm{GL}_n)^{(1)}$ -representation V . For $r = 1$, we already introduced the twist

$$\Omega_1^\bullet \otimes_l P_1^* V \cong \mathcal{I}_1(\Lambda^\bullet U \otimes V^{[1]})$$

By Cartier's Theorem, its cohomology computes as

$$H^i(\Omega_1^\bullet) \otimes_k P_1^* V \cong P_1^*(\Lambda^i U^{(1)} \otimes_k V)$$

Again, we consider the functor

$$\operatorname{ind}_{U_1}^{G(n, r)} \circ t_{r,1}^* : G(n, 1)\text{-rep} \longrightarrow G(n, r)\text{-rep}$$

According to Lemma 4.3, it provides a complex

$$\operatorname{ind}_{U_1}^{G(n, r)}(t_{r,1}^* \mathcal{I}_1(\Lambda^\bullet U \otimes V^{[1]})) = \mathcal{I}_r(\Lambda^\bullet U \otimes V^{[1]})$$

of $G(n, r)$ -representations. As GL_n -representations, this complex reads as

$$\Omega_r^{i-1} \otimes_k V^{[1]} \xrightarrow{d_i \otimes \operatorname{id}} \Omega_r^i \otimes_k V^{[1]}$$

Similarly as in the previous Corollary, we obtain

$$H^i(\Omega_r^\bullet \otimes_k V^{[1]}) \cong T_r^*((\Omega_{r-1}^i)^{(1)} \otimes_k V)$$

for its cohomology.

6. IRREDUCIBLE REPRESENTATIONS

We are now going to compute the irreducible $G(n, r)$ -representations

$$L(\lambda, G(n, r)) = \operatorname{soc} \mathcal{I}(L(\lambda)) = G(n, r)L(\lambda) \subset \mathcal{I}(L(\lambda))$$

for all $\lambda \in X(T)_+$ with respect to their associated irreducible GL_n -representations. For this, we take as split maximal torus the diagonal matrices. This torus affords canonical projections $\varepsilon_i \in X(T)$ for $1 \leq i \leq n$ which are a free \mathbb{Z} -basis of the character group $X(T)$.

According to Proposition 3.13 there is a mod p^r -periodicity for the dominant weights and one can restrict to the $L(\lambda, G(n, r))$ with $\lambda \in X_r(T)$. This will cover the case $r = 1$. The case $r \geq 2$ is more subtle.

We restrict to the following subset $X'_1(T) \subset X_1(T)$:

$$X'_1(T) := \left\{ \lambda = \sum_{i=1}^n m_i(\epsilon_1 + \dots + \epsilon_i) \in X(T) \mid \forall 1 \leq i \leq n : 0 \leq m_i < p \right\}$$

Notation 6.1. As $X'_1(T)$ is a set of representatives for $X(T)/pX(T)$, we get a unique decomposition

$$\lambda = r(\lambda) + ps(\lambda)$$

for all $\lambda \in X(T)_+$ with $r(\lambda) \in X'_1(T)$ and $s(\lambda) \in X(T)_+$. We call $r(\lambda)$ the mod p -reduction of λ .

The following Proposition covers the dominant weights λ with $r(\lambda) = 0$.

Proposition 6.2. *Let $\lambda \in X(T)_+$. Then we obtain*

$$L(p\lambda, G(n, 1)) \cong P_1^* L(\lambda)$$

and for $r \geq 2$ we get

$$L(p\lambda, G(n, r)) \cong T_r^* L(\lambda, G(n, r-1))^{(1)}$$

Proof. The claim for $r = 1$ is just a special case of Proposition 3.13 and the claim for $r \geq 2$ follows by the discussion after Lemma 4.1. \square

The idea for the case $r(\lambda) \neq 0$ is the following: The $G(n, r)^-$ -invariants of the socle of $\mathcal{I}(L(\lambda)) = L(\lambda) \otimes R(n, r)$ are $L(\lambda) \otimes k \cong L(\lambda)$. That is, the socle is generated by this subspace as a $G(n, r)$ -representation.

Notation 6.3. For $\lambda \in X'_1(T)$ write $\lambda = \sum_{i=1}^n m_i(\epsilon_1 + \dots + \epsilon_i)$ where $0 \leq m_i < p$ and consider the GL_n -representation

$$\mathrm{Sym}^{m_1}(U) \otimes_k \mathrm{Sym}^{m_2}(\Lambda^2 U) \otimes_k \dots \otimes_k \mathrm{Sym}^{m_n}(\Lambda^n U)$$

where $U = k^n$ with canonical basis e_1, \dots, e_n . Now consider the vector

$$v(\lambda) = e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \dots \otimes (e_1 \wedge \dots \wedge e_n)^{m_n}$$

in this representation. We define $W(\lambda)$ to be the GL_n -subrepresentation generated by this vector.

For a general $\lambda \in X(T)_+$ set

$$W(\lambda) := W(r(\lambda)) \otimes L(s(\lambda))^{[1]}$$

Note that $W(r(\lambda))$ has highest weight $r(\lambda)$ of multiplicity 1. That is, there is a subrepresentation $V \subset W(r(\lambda))$ such that

$$L(r(\lambda)) \cong W(r(\lambda))/V$$

Thus

$$L(\lambda) \cong W(\lambda)/(V \otimes_k L(s(\lambda))^{[1]})$$

by Steinberg's Tensor Product Theorem. For example if $\lambda = \epsilon_1 + \dots + \epsilon_i$ is a fundamental weight, then

$$L(\epsilon_1 + \dots + \epsilon_i) = \Lambda^i U = W(\lambda)$$

As \mathcal{I} is exact, we get

$$\mathcal{I}(L(\lambda)) = \mathcal{I}(W(\lambda))/\mathcal{I}(V \otimes_k L(s(\lambda))^{[1]})$$

Since $v(r(\lambda))$ is a highest weight vector, it follows that

$$\begin{aligned} \text{soc } \mathcal{I}(L(\lambda)) &= G(n, r)L(\lambda) \\ &= G(n, r)(v(r(\lambda)) \otimes_k L(s(\lambda))^{[1]}) / \mathcal{I}(V \otimes_k L(s(\lambda))^{[1]}) \end{aligned}$$

Finally, note that

$$\begin{aligned} &\mathcal{I}(\text{Sym}^{m_1}(U) \otimes_k \text{Sym}^{m_2}(\Lambda^2 U) \otimes_k \dots \otimes_k \text{Sym}^{m_n}(\Lambda^n U)) \\ &= R(n, r) \otimes_k \text{Sym}^{m_1}(U) \otimes_k \text{Sym}^{m_2}(\Lambda^2 U) \otimes_k \dots \otimes_k \text{Sym}^{m_n}(\Lambda^n U) \\ &\cong \text{Sym}_{R(n, r)}^{m_1}(\Omega_r^1) \otimes_{R(n, r)} \text{Sym}_{R(n, r)}^{m_2}(\Omega_r^2) \otimes_{R(n, r)} \dots \otimes_{R(n, r)} \text{Sym}_{R(n, r)}^{m_n}(\Omega_r^n) \end{aligned}$$

as $G(n, r)$ -representations and for $\lambda = r(\lambda)$, the vector $v(\lambda)$ corresponds to

$$v = (dx_1)^{m_1} \otimes (dx_1 \wedge dx_2)^{m_2} \otimes \dots \otimes (dx_1 \wedge \dots \wedge dx_n)^{m_n}$$

In order to compute $G(n, r)v \subset \mathcal{I}(W(\lambda))$, we will use the Lie algebra operators $\delta_{(i, x^I)}$. The following Proposition describes their action on $\mathcal{I}(V)$.

Lemma 6.4. *Let $\delta_{(i, x^I)} \in \text{Lie } G(n, r)$ be a canonical basis element. Then for all GL_n -representations V , the induced action on*

$$\mathcal{I}(V) = R(n, r) \otimes_k V$$

reads as

$$\delta_{(i, x^I)}(f \otimes v) = \left(x^I \frac{\partial}{\partial x_i} f \right) \otimes v + \sum_{j=1}^n f \frac{\partial}{\partial x_j} x^I \otimes E_{ji}(v)$$

where $E_{ji} \in M_n(k) = \text{Lie } \text{GL}_n$ is the (j, i) -th standard matrix.

Proof. First recall the $G(n, r)$ -action on $\mathcal{I}(V) = R(n, r) \otimes_k V$:

$$g(f \otimes v) = \left(\frac{\partial g(x_s)}{\partial x_k} \right)_{ks} (g(f) \otimes v)$$

for all $g \in G(n, r)$, $f \in R(n, r)$, and $v \in V$.

In order to compute the action of $\delta_{(i, x^I)} = x^I \frac{\partial}{\partial x_i}$, we consider the corresponding element $g_{(i, I)} = 1 + \delta_{(i, x^I)} \epsilon \in G(n, r)(k[\epsilon])$ where $k[\epsilon]$ are the dual numbers. That is,

$$g_{(i, I)}(x_s) = \begin{cases} x_s + x^I \epsilon & s = i \\ x_s & s \neq i \end{cases}$$

Now we get

$$\delta_{(i, x^I)}(f \otimes v) = \frac{\partial}{\partial \epsilon} \left(\left(\frac{\partial g_{(i, I)}(x_s)}{\partial x_k} \right)_{ks} (g_{(i, I)}(f) \otimes v) \right) \Big|_{\epsilon=0}$$

The product rule provides

$$\delta_{(i, x^I)}(f \otimes v) = \delta_{(i, x^I)}(f) \otimes v + f \frac{\partial}{\partial \epsilon} \left(\left(\frac{\partial g_{(i, I)}(x_s)}{\partial x_k} \right)_{ks} (1 \otimes v) \right) \Big|_{\epsilon=0}$$

As

$$\frac{\partial g_{(i, I)}(x_s)}{\partial x_k} = \begin{cases} 1 + \frac{\partial x^I}{\partial x_k} \epsilon & s = i = k \\ \frac{\partial x^I}{\partial x_k} \epsilon & s = i \neq k \\ 1 & s = k \neq i \\ 0 & s \neq k, s \neq i \end{cases}$$

we get

$$\left. \frac{\partial}{\partial \epsilon} \left(\frac{\partial}{\partial x_k} g_{(i, I)}(x_s) \right) \right|_{\epsilon=0} = \begin{cases} \frac{\partial x^I}{\partial x_k} & s = i \\ 0 & s \neq i \end{cases}$$

Whence the claim. \square

Remark 6.5. Note that the action of $\delta_{(i, x^I)}$ on $W(\lambda) = W(r(\lambda)) \otimes L(s(\lambda))^{[1]}$ is $(-) \otimes_k \text{id}_{L(s(\lambda))^{[1]}}$ applied to the action on $W(r(\lambda))$ as $\text{Lie } \text{GL}_n$ acts trivially on Frobenius twists $V^{[1]}$.

Now we are ready to treat the case where the mod p -reduction of λ is a fundamental weight $\varepsilon_1 + \dots + \varepsilon_i$. By Steinberg's Tensor Product Theorem, we know that

$$L(\lambda) \cong L(\epsilon_1 + \dots + \epsilon_i) \otimes_k L(s(\lambda))^{[1]} \cong \Lambda^i U \otimes_k L(s(\lambda))^{[1]} = W(\lambda)$$

with $U = k^n$. That is,

$$\mathcal{I}(L(\lambda)) \cong \Omega_r^i \otimes_k L(s(\lambda))^{[1]}$$

which is part of the twisted de Rham complex as introduced at the end of the previous section. Recall that the differentials read as

$$\Omega_r^{i-1} \otimes_k L(s(\lambda))^{[1]} \xrightarrow{d_i \otimes \text{id}} \Omega_r^i \otimes_k L(s(\lambda))^{[1]}$$

where $d_i : \Omega_r^{i-1} \rightarrow \Omega_r^i$ is the de Rham-differential.

Proposition 6.6. *Let $\lambda \in X(T)_+$ with $r(\lambda) = \epsilon_1 + \dots + \epsilon_i$, then*

$$L(\lambda, G(n, r)) \cong \text{soc}(\Omega_r^i \otimes_k L(s(\lambda))^{[1]}) = \text{Im}(d_i \otimes \text{id})$$

where $d_i : \Omega_r^{i-1} \rightarrow \Omega_r^i$ is the de Rham-differential.

Proof. We will use $\text{Lie}(G(n, r))$ -operators to prove the claim. Recall that $f \in \text{Lie } G(n, r)$ acts on $\Omega_r^i \otimes_k L(s(\lambda))^{[1]}$ as $f \otimes \text{id}$.

We already noticed that the socle is generated by the $G(n, r)^-$ -invariants as a $G(n, r)$ -representation. A generating system of these invariants is given by

$$(dx_{j_1} \wedge \dots \wedge dx_{j_i}) \otimes v$$

for all $j_1 < \dots < j_i$ and $v \in L(s(\lambda))^{[1]}$. Now the inclusion

$$\text{soc}(\Omega_r^i \otimes L(s(\lambda))^{[1]}) \subset \text{Im}(d_i \otimes \text{id})$$

follows from the fact that the generators lie in the image of $d_i \otimes \text{id}$. For the inclusion $\text{Im}(d_i \otimes \text{id}) \subset \text{soc}(\Omega_r^i \otimes L(s(\lambda))^{[1]})$ note that $\text{Im}(d_i \otimes \text{id})$ is as a k -vector space generated by the elements

$$(dx^I \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{i-1}}) \otimes v$$

where $v \in L(s(\lambda))^{[1]}$, $x^I = x_1^{m_1} \dots x_n^{m_n} \in R(n, r)$, and $j_1 < \dots < j_{i-1}$. As $i - 1 < n$, there is an index $l \notin \{j_1, \dots, j_{i-1}\}$. Then we get that the Lie algebra operator $\delta_{(l, x^I)} \in \text{Lie}(G(n, r))$ acts as

$$\delta_{(l, x^I)}((dx_l \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{i-1}}) \otimes v) = (dx^I \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{i-1}}) \otimes v$$

according to Lemma 6.4. This provides all image elements from the generators. \square

The next Proposition covers the case where the mod p -reduction of λ is neither 0 nor a fundamental weight if we assume $\text{char}(k) \neq 2$.

Proposition 6.7. *Assume that $\text{char}(k) \neq 2$. Let $\lambda \in X(T)_+$ a dominant weight with $r(\lambda) \neq 0$ and $r(\lambda) \neq \epsilon_1 + \dots + \epsilon_i$ for all $i = 1, \dots, n$. Then*

$$L(\lambda, G(n, r)) = \mathcal{I}(L(\lambda))$$

Before we give the proof, we need some technical Lemmas.

Lemma 6.8. *Let V be GL_n -representation, $v \in V$, and $1 \leq s_j \leq p^{r-1}$. If*

$$x_1^{ps_1-1} \dots x_n^{ps_n-1} v \in G(n, r)v \subset R(n, r) \otimes_k V = \mathcal{I}(V)$$

then

$$x^J v \in G(n, r)v \subset \mathcal{I}(V)$$

for all $J = (j_1, \dots, j_n)$ with $p(s_k - 1) \leq j_k < ps_k$ for all $1 \leq k \leq n$.

Proof. This follows by the gradual application of the Lie algebra operators

$$\delta_i = \frac{\partial}{\partial x_i} \otimes \text{id} : R(n, r) \otimes V \rightarrow R(n, r) \otimes V \quad \square$$

Lemma 6.9. *Let V be a GL_n -representation, $v \in V$, and $1 \leq s_j \leq p^{r-1}$. Assume that $x^J v \in G(n, r)v \subset R(n, r) \otimes_k V = \mathcal{I}(V)$ for $J = (j_1, \dots, j_n)$.*

(1) *If $j_k = ps$, then for all $j \neq k$ we get*

$$x^J E_{jk} v \in G(n, r)v \subset \mathcal{I}(V)$$

(2) *If $j_i = ps$ and $x_j \frac{\partial}{\partial x_k} x^J v \in G(n, r)v$ we get*

$$x^J E_{ki} E_{jk} v \in G(n, r)v \subset \mathcal{I}(V)$$

Proof. The first part follows from

$$x^J E_{jk} v = \delta_{(k, x_j)}(x^J v)$$

as $j_k = ps$.

The second part follows from

$$\begin{aligned} x^J E_{ki} E_{jk} v &= \delta_{(i, x_k)}(x^J E_{jk} v) \\ &= \delta_{(i, x_k)} \left(\delta_{(k, x_j)}(x^J v) - x_j \frac{\partial}{\partial x_k} x^J v \right) \end{aligned}$$

as $j_i = ps$. \square

Lemma 6.10. *Let $\lambda \in X'_1(T)$ and write $\lambda = \sum_{i=1}^n m_i(\epsilon_1 + \dots + \epsilon_i)$. Let $v = v(\lambda) \in W(\lambda)$ the vector from above. Let k be the highest index, such that $m_k \neq 0$ and $i < k$ the highest index such that $m_i \neq 0$.*

(1) *For all $j \leq k$, we get*

$$E_{jk} v = \delta_{jk} m_k v$$

where δ_{jk} is the Kronecker- δ .

(2) *For $j > k$, we get*

$$E_{kk} E_{jk} v = (m_k - 1) E_{jk} v$$

(3) *If $m_k = 1$, we get*

$$E_{ii} v = (m_i + 1) v$$

(4) If $m_k = 1$, we get for all $j > k$

$$E_{ii}E_{ki}v = m_iE_{ki}v$$

and

$$E_{ii}E_{ki}E_{jk}v = m_iE_{ki}E_{jk}v$$

Proof. Recall that

$$v = e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \dots \otimes (e_1 \wedge \dots \wedge e_k)^{m_k}$$

The first claim follows by

$$E_{jk}v = m_k e_1^{m_1} \otimes (e_1 \wedge e_2)^{m_2} \otimes \dots \otimes (e_1 \wedge \dots \wedge e_{k-1} \wedge e_j)(e_1 \wedge \dots \wedge e_k)^{m_k-1}$$

The second follows from the first by

$$E_{kk}E_{jk}v = E_{jk}E_{kk}v + (E_{kk}E_{jk} - E_{jk}E_{kk})v,$$

using $E_{kk}E_{jk} = 0$ and $E_{jk}E_{kk} = E_{jk}$.

Now let $m_k = 1$. Then the third claim follows as E_{ii} acts precisely on the factors $(e_1 \wedge \dots \wedge e_i)^{m_i}$ and $(e_1 \wedge \dots \wedge e_k)$ of v .

The claim $E_{ii}E_{ki}v = m_iE_{ki}v$ follows from the third in the same fashion as the second follows from the first.

Finally for the last claim using $E_{ii}E_{jk} = E_{jk}E_{ii} = 0$ and the third claim, we get

$$E_{ii}E_{jk}v = E_{jk}E_{ii}v = (m_i + 1)E_{jk}v$$

Hence

$$\begin{aligned} E_{ii}E_{ki}E_{jk}v &= E_{ki}E_{ii}E_{jk}v - E_{ki}E_{jk}v \\ &= m_iE_{ki}E_{jk}v \end{aligned}$$

which finishes the proof. \square

Proof of 6.7. By previous discussions, it suffices to prove

$$G(n, r)(v(r(\lambda)) \otimes_k L(s(\lambda))^{[1]}) = \mathcal{I}(W(r(\lambda)) \otimes_k L(s(\lambda))^{[1]}) = \mathcal{I}(W(\lambda))$$

Again we will use $\text{Lie}(G(n, r))$ -operators to prove the claim. As again $f \in \text{Lie } G(n, r)$ acts as $f \otimes \text{id}$, we can assume that $\lambda = r(\lambda)$.

Let $v = v(\lambda)$ as above. It suffices to show

$$R(n, r) \otimes_k kv(\lambda) \subset G(n, r)v(\lambda) \subset \mathcal{I}(W(\lambda))$$

Now write again

$$\lambda = \sum_{i=1}^n m_i(\epsilon_1 + \dots + \epsilon_i)$$

As $\lambda = r(\lambda)$, we have $0 \leq m_i \leq p-1$ for all $i = 1, \dots, n$.

By Lemma 6.8, it suffices to prove

$$x_1^{ps_1-1} \dots x_n^{ps_n-1}v \in G(n, r)v$$

for all choices $1 \leq s_j \leq p^{r-1}$.

By assumption we have $\lambda \neq 0$. That is, there is a highest index k such that $m_k \neq 0$.

Case 1 ($m_k \geq 2$). Let us assume that $m_k \geq 2$. We argue by descending induction on s_k .

Take $I = (ps_1 - 1, \dots, ps_n - 1)$. Note that for all $J = (j_1, \dots, j_n)$ with $j_k \geq ps_k$ we get

$$(1) \quad x^J v \in G(n, r)v$$

For $s_k = p^{r-1}$ this is clear as in this case $x^J = 0$. The case $s_k < p^{r-1}$ follows by the induction hypothesis and Lemma 6.8.

By Lemma 6.4 we get

$$(2) \quad \delta_{(k, x^I)}(v) = m_k \frac{\partial}{\partial x_k} x^I v + \sum_{j > k} \frac{\partial}{\partial x_j} x^I E_{jk} v$$

since $E_{jk} v = \delta_{jk} m_k v$ for $j \leq k$ by Lemma 6.10(1).

Now we apply the operator $\delta_{(k, x_k^2)}$ to $\delta_{(k, x^I)}(v)$. Let $j > k$. Using $E_{kk} E_{jk} v = (m_k - 1) E_{jk} v$ Lemma 6.10(2) we obtain

$$\begin{aligned} \delta_{(k, x_k^2)} \left(\frac{\partial}{\partial x_j} x^I E_{jk} v \right) &= x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} x^I E_{jk} v + 2x_k \frac{\partial}{\partial x_j} x^I E_{kk} E_{jk} v \\ &= x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} x^I E_{jk} v + 2(m_k - 1)x_k \frac{\partial}{\partial x_j} x^I E_{jk} v \\ &\in G(n, r)v \end{aligned}$$

by (1) and Lemma 6.9(1).

As $\delta_{(k, x_k^2)}(\delta_{(k, x^I)}(v)) \in G(n, r)v$, we obtain

$$\delta_{(k, x_k^2)} \left(\frac{\partial}{\partial x_k} x^I v \right) \in G(n, r)v$$

by (2). Using $E_{kk} v = m_k v$, we get

$$\begin{aligned} \delta_{(k, x_k^2)} \left(\frac{\partial}{\partial x_k} x^I v \right) &= x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} x^I v + 2x_k \frac{\partial}{\partial x_k} x^I E_{kk} v \\ &= \left(x_k^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} x^I + 2m_k x_k \frac{\partial}{\partial x_k} x^I \right) v \\ &= (ps_k - 1)(ps_k - 2 + 2m_k) x^I v \\ &= 2(1 - m_k) x^I v \\ &\in G(n, r)v \end{aligned}$$

As $2 \leq m_k \leq p - 1$ and $\text{char}(k) = p \neq 2$, we have $2(1 - m_k) \neq 0$. Hence

$$x_1^{ps_1-1} \dots x_n^{ps_n-1} v = x^I v \in G(n, r)v$$

for all choices $1 \leq s_j \leq p^{r-1}$ which finishes the proof for the case $m_k \geq 2$.

Case 2 ($m_k = 1$). In the second case, we assume that $m_k = 1$. As $r(\lambda) = \lambda$ is not a fundamental weight by assumption, there is a highest index $i < k$ with $m_i \neq 0$. Here, we will argue by descending induction on $s_i + s_k$.

Take again $I = (ps_1 - 1, \dots, ps_n - 1)$. Note that for all $J = (j_1, \dots, j_n)$ with $j_k \geq ps_k \wedge j_i \geq p(s_i - 1)$ or $j_k \geq p(s_k - 1) \wedge j_i \geq ps_i$, we get

$$(3) \quad x^J v \in G(n, r)v$$

For $s_k = p^{r-1}$ or $s_i = p^{r-1}$, this is clear as in this case $x^J = 0$. In the case $j_k \geq ps_k$ and $j_i \geq p(s_i - 1)$ and $s_k < p^{r-1}$ it follows by the induction hypothesis and Lemma 6.8. The case $j_k \geq p(s_k - 1)$ and $j_i \geq ps_i$ is analogous.

Again by Lemma 6.4, we obtain

$$(4) \quad \delta_{(k, x^I)}(v) = \frac{\partial}{\partial x_k} x^I v + \sum_{j > k} \frac{\partial}{\partial x_j} x^I E_{jk} v$$

as $E_{kk} v = m_k v = v$.

For $j > k$ we get

$$x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{jk} v \in G(n, r) v$$

by (3) and Lemma 6.9(1).

Now we apply the operator $\delta_{(i, x_i^2)} \circ \delta_{(i, x_k)}$ to $\delta_{(k, x^I)}(v)$. For $j > k$, we get

$$\begin{aligned} & \delta_{(i, x_i^2)} \left(\delta_{(i, x_k)} \left(\frac{\partial}{\partial x_j} x^I E_{jk} v \right) \right) \\ &= \underbrace{\delta_{(i, x_i^2)} \left(x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{jk} v \right)}_{\in G(n, r) v} + \delta_{(i, x_i^2)} \left(\frac{\partial}{\partial x_j} x^I E_{ki} E_{jk} v \right) \end{aligned}$$

But using $E_{ii} E_{ki} E_{jk} v = m_i E_{ki} E_{jk} v$ Lemma 6.10(4), we get for $j > k$

$$\begin{aligned} \delta_{(i, x_i^2)} \left(\frac{\partial}{\partial x_j} x^I E_{ki} E_{jk} v \right) &= x_i^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{ki} E_{jk} v + 2x_i \frac{\partial}{\partial x_j} x^I E_{ii} E_{ki} E_{jk} v \\ &= x_i^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} x^I E_{ki} E_{jk} v + 2m_i x_i \frac{\partial}{\partial x_j} x^I E_{ki} E_{jk} v \\ &\in G(n, r) v \end{aligned}$$

by (3) and Lemma 6.9(2).

As $\delta_{(i, x_i^2)}(\delta_{(i, x_k)}(\delta_{(k, x^I)}(v))) \in G(n, r) v$, we obtain

$$\delta_{(i, x_i^2)} \left(\delta_{(i, x_k)} \left(\frac{\partial}{\partial x_k} x^I v \right) \right) \in G(n, r) v$$

by (4). That is, by using

$$x_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I = (ps_k - 1) \frac{\partial}{\partial x_i} x^I = -\frac{\partial}{\partial x_i} x^I$$

we get

$$\delta_{(i, x_i^2)} \left(-\frac{\partial}{\partial x_i} x^I v + \frac{\partial}{\partial x_k} x^I E_{ki} v \right) \in G(n, r) v$$

Using $E_{ii} E_{ki} v = m_i E_{ki} v$ Lemma 6.10(4), we obtain

$$\begin{aligned} \delta_{(i, x_i^2)} \left(\frac{\partial}{\partial x_k} x^I E_{ki} v \right) &= x_i^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I E_{ki} v + 2x_i \frac{\partial}{\partial x_k} x^I E_{ii} E_{ki} v \\ &= x_i^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} x^I E_{ki} v + 2m_i x_i \frac{\partial}{\partial x_k} x^I E_{ki} v \\ &\in G(n, r) v \end{aligned}$$

by (3) and Lemma 6.9(1). Hence

$$\delta_{(i, x_i^2)} \left(\frac{\partial}{\partial x_i} x^I v \right) \in G(n, r)v$$

Using $E_{ii}v = (m_i + 1)v$ Lemma 6.10(3), we obtain

$$\begin{aligned} \delta_{(i, x_i^2)} \left(\frac{\partial}{\partial x_i} x^I v \right) &= x_i^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} x^I v + 2x_i \frac{\partial}{\partial x_i} x^I E_{ii}v \\ &= 2x^I v - 2(m_i + 1)x^I v \\ &= -2m_i x^I v \\ &\in G(n, r)v \end{aligned}$$

That is, we get

$$2m_i x^I v \in G(n, r)v$$

which is nonzero as $\text{char}(k) = p \neq 2$ and $1 \leq m_i \leq p - 1$. Hence

$$x_1^{ps_1-1} \dots x_n^{ps_n-1} v = x^I v \in G(n, r)v$$

which finishes the proof for $m_k = 1$.

Finally, we proved the Proposition. □

7. THE GROTHENDIECK RING

For a k -group scheme G denote by

$$\text{Rep}(G) := K_0(G\text{-rep})$$

the Grothendieck ring of finite dimensional representations. This is a free abelian group where a \mathbb{Z} -basis is given by the classes of the irreducible representations.

Using the description of irreducible $G(n, r)$ -representations of the previous section, we are now ready to describe $\text{Rep}(G(n, r))$. Recall that the functor \mathcal{I} is exact.

Theorem 7.1. *Assume that $\text{char}(k) \neq 2$. Then the maps*

$$\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(G(n, 1))$$

and for $r \geq 2$

$$\begin{pmatrix} \mathcal{I} \\ T_r^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) \rightarrow \text{Rep}(G(n, r))$$

are surjective morphisms of abelian groups.

Proof. Recall that a \mathbb{Z} -basis of $\text{Rep}(G(n, r))$ is given by $[L(\lambda, G(n, r))]$ with $\lambda \in X(T)_+$. That is, it suffices to show that these classes lie in the respective image.

We start with the case $r = 1$ and the map $\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix}$. For $r(\lambda) = 0$, we know by Proposition 6.2 that

$$L(\lambda, G(n, 1)) = P_1^* L(s(\lambda))$$

which shows the claim. For $r(\lambda) = \varepsilon_1 + \dots + \varepsilon_i$, we know by Proposition 6.6 that

$$L(\lambda, G(n, 1)) \cong \text{Im}(d_i \otimes \text{id} : \Omega_1^{i-1} \otimes_k L(s(\lambda))^{[1]} \rightarrow \Omega_1^i \otimes_k L(s(\lambda))^{[1]})$$

the images in the twisted de Rham complex. We also know that its cohomology computes as

$$H^i(\Omega_1^\bullet \otimes_k L(s(\lambda))^{[1]}) \cong P_1^*(\Lambda^i U^{(1)} \otimes_k L(s(\lambda)))$$

That is, the cohomology classes lie in the image of $\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix}$. As

$$\Omega_1^i \otimes_k L(s(\lambda))^{[1]} = \mathcal{I}(\Lambda^i U \otimes_k L(s(\lambda))^{[1]})$$

also the classes of the objects of the complex lie in this image. As

$$[\text{Im}(d_n \otimes \text{id})] = [\Omega_1^n \otimes_k L(s(\lambda))^{[1]}] + [H^n(\Omega_1^\bullet \otimes_k L(s(\lambda))^{[1]})]$$

we get the claim for $r(\lambda) = \varepsilon_1 + \dots + \varepsilon_n$. Now let $i < n$. Then

$$\begin{aligned} & [\text{Im}(d_i \otimes \text{id})] \\ = & [\Omega_1^i \otimes_k L(s(\lambda))^{[1]}] - [\text{Im}(d_{i+1} \otimes \text{id})] - [H^i(\Omega_1^\bullet \otimes_k L(s(\lambda))^{[1]})] \end{aligned}$$

inductively provides the claim for $r(\lambda) = \varepsilon_1 + \dots + \varepsilon_i$. Finally, in the case that $r(\lambda)$ is neither 0 nor a fundamental weight, Proposition 6.7 provides

$$L(\lambda, G(n, r)) = \mathcal{I}(L(\lambda))$$

which finishes the case $r = 1$.

Now let $r \geq 2$. That is, we consider the map $\begin{pmatrix} \mathcal{I} \\ T_r^* \end{pmatrix}$. Then for $r(\lambda) = 0$, we get

$$L(\lambda, G(n, r)) = T_r^* L(s(\lambda), G(n, r-1))^{(1)}$$

by Proposition 6.2 which shows the claim. For the case $r(\lambda) = \varepsilon_1 + \dots + \varepsilon_i$ we use the twisted de Rham complex

$$\Omega_r^i \otimes_k L(s(\lambda))^{[1]} = \mathcal{I}(\Lambda^i U \otimes_k L(s(\lambda))^{[1]})$$

As the image of the i -th differential is $L(\lambda, G(n, r))$ by Proposition 6.6 and its cohomology computes as

$$H^i(\Omega_r^\bullet \otimes_k L(s(\lambda))^{[1]}) \cong T_r^*((\Omega_{r-1}^i)^{(1)} \otimes_k L(s(\lambda)))$$

the claim follows in the same fashion as in the case $r = 1$. Again the case where $r(\lambda)$ is neither 0 nor fundamental follows from Proposition 6.7. \square

Remark 7.2. Let $\text{char}(k) = 2$. Then for $n = r = 1$, the case where $r(\lambda)$ is neither 0 nor fundamental does not occur. But this is the only case where the assumption $\text{char}(k) \neq 2$ is necessary in order to apply Proposition 6.7. That is, for $n = r = 1$, we also get a surjection

$$\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix} : \text{Rep}(\mathbb{G}_m) \oplus \text{Rep}((\mathbb{G}_m)^{(1)}) \rightarrow \text{Rep}(G(1, 1))$$

In the case that $r \geq 2$ or $n \geq 2$, the author does not know whether the maps in question are surjective.

So far, we only described the abelian group structure of $\text{Rep}(G(n, r))$. We also want to understand its ring structure and the kernels of the maps of the theorem. For both topics, the main tool is the restriction map

$$\text{res} : \text{Rep}(G(n, r)) \rightarrow \text{Rep}(\text{GL}_n)$$

In fact, this map is injective.

Lemma 7.3. *The restriction map*

$$\text{res} : \text{Rep}(G(n, r)) \rightarrow \text{Rep}(\text{GL}_n)$$

is injective.

Proof. As $\mathbb{G}_m \subset \text{GL}_n \subset G(n, r)$, we can study \mathbb{G}_m -weight spaces for $G(n, r)$ -representations. Furthermore, the \mathbb{G}_m -weight space filtration of each GL_n -representation is GL_n -invariant. Denote by $\deg : X(T) \rightarrow \mathbb{Z}$ the degree map which is induced by $\deg(\epsilon_i) = 1$ for all $i = 1, \dots, n$. Then for all $n \in \mathbb{Z}$ and GL_n -representations V , we get for the n -th \mathbb{G}_m -weight space

$$V_n = \bigoplus_{\lambda \in \deg^{-1}(n)} V_\lambda$$

Now consider $L(\lambda, G(n, r))$ for $\lambda \in X(T)_+$, whose classes form a \mathbb{Z} -basis of $\text{Rep}(G(n, r))$. Its lowest \mathbb{G}_m -weight space is $L(\lambda)$ of weight $s = \deg(\lambda)$. Hence, we obtain

$$[\text{res } L(\lambda, G(n, r))] = [L(\lambda)] + \sum_{\substack{\mu \in X(T)_+ \\ \deg(\mu) > s}} m_\mu [L(\mu)] \in \text{Rep}(\text{GL}_n)$$

where m_μ is the multiplicity of $L(\mu)$ in $L(\lambda, G(n, r))$. As $[L(\lambda)]$ with $\lambda \in X(T)_+$ form a \mathbb{Z} -basis of $\text{Rep}(\text{GL}_n)$, res maps a \mathbb{Z} -basis of $\text{Rep}(G(n, r))$ to a linearly independent set. Hence res is injective. \square

Now $\text{Rep}(\text{GL}_n)$ is a $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra by the pullback of the first Frobenius

$$F^* : \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

Note that under the identification $\text{GL}_n^{(1)} \cong \text{GL}_n$ the ring homomorphism F^* is the p -th Adams operation ψ^p on the λ -ring $\text{Rep}(\text{GL}_n)$. We fix an r and endow the direct sum

$$\text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})$$

with a ring structure by assigning

$$(b, a) \cdot (b', a') := (F^*(a)b' + F^*(a')b + [R(n, r)]bb', aa')$$

where $[R(n, r)]$ is the class of the $G(n, r)$ -representation $R(n, r)$ restricted to the subgroup GL_n . Now the inclusion

$$\text{Rep}((\text{GL}_n)^{(1)}) \hookrightarrow \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})$$

gives an $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra structure. Furthermore the map

$$\left(\begin{smallmatrix} [R(n, r)] \cdot (-) \\ F^* \end{smallmatrix} \right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

is an $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra morphism. Finally note that

$$\text{res} \circ \mathcal{I} = [R(n, r)] \cdot (-)$$

on $\text{Rep}(\text{GL}_n)$.

Let us consider the case $r = 1$. In order to show that

$$\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(G(n, 1))$$

is a ring homomorphism, it suffices to show this after composition with res . But this is precisely the ring homomorphism $\begin{pmatrix} [R(n, 1)] \cdot (-) \\ F^* \end{pmatrix}$ considered above.

Now consider the case $r \geq 2$. Then we introduce a ring structure on

$$\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)})$$

by

$$(b, a) \cdot (b', a') := (F^*(\text{res}(a))b' + F^*(\text{res}(a'))b + [R(n, r)]bb', aa')$$

where we use the restriction

$$\text{res} : \text{Rep}(G(n, r-1)^{(1)}) \hookrightarrow \text{Rep}((\text{GL}_n)^{(1)})$$

This makes

$$\text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) \xrightarrow{\text{id} \oplus \text{res}} \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})$$

into a ring injection. In order to show that

$$\begin{pmatrix} \mathcal{I} \\ T_r^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) \rightarrow \text{Rep}(G(n, r))$$

is a ring homomorphism, we compose with res again. But this composition coincides with the ring injection $\text{id} \oplus \text{res}$ followed by the ring homomorphism

$$\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

considered above.

Now we come to kernel elements. For all $r \geq 0$, let us consider the element

$$\delta_r := \sum_{i=0}^n (-1)^{n-i} [\Lambda^i U^{(r)}] \in \text{Rep}((\text{GL}_n)^{(r)})$$

where $U = k^n$ and for $r \geq 1$ the $r-1$ -th Frobenius

$$F^{r-1} : (\text{GL}_n)^{(1)} \rightarrow (\text{GL}_n)^{(r)}$$

Proposition 7.4. *For all $r \geq 1$, the kernel of*

$$\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

is generated by $(\delta_0, -(F^{r-1})^(\delta_r))$ as an $\text{Rep}((\text{GL}_n)^{(1)})$ -module.*

For $r = 1$, $\text{Rep}(G(n, 1))$ is a $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra by P_1^* . Then

$$\begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix} : \text{Rep}(\text{GL}(U)) \oplus \text{Rep}(\text{GL}(U)^{(1)}) \rightarrow \text{Rep}(G(n, 1))$$

is an $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra map. The Proposition implies that its kernel is generated by $(\delta_0, -\delta_1)$ as an $\text{Rep}((\text{GL}_n)^{(1)})$ -module as

$$\text{res} \circ \begin{pmatrix} \mathcal{I} \\ P_1^* \end{pmatrix} = \begin{pmatrix} [R(n, 1)] \cdot (-) \\ F^* \end{pmatrix}$$

and $\text{res} : \text{Rep}(G(n, 1)) \rightarrow \text{Rep}(\text{GL}_n)$ is an injective $\text{Rep}((\text{GL}_n)^{(1)})$ -algebra map.

For $r \geq 2$, we do not even have a $\text{Rep}((\text{GL}_n)^{(1)})$ -module structure on $\text{Rep}(G(n, r))$. But we can study the commutative diagram

$$\begin{array}{ccc} \text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) & \xrightarrow{\begin{pmatrix} \mathcal{I} \\ T_r^* \end{pmatrix}} & \text{Rep}(G(n, r)) \\ \text{id} \oplus \text{res} \downarrow & & \downarrow \text{res} \\ \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) & \xrightarrow{\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix}} & \text{Rep}(\text{GL}_n) \end{array}$$

and get the following.

Corollary 7.5. *For all $r \geq 2$, the image*

$$(\text{id} \oplus \text{res})(\text{Ker}(\mathcal{I} + T_r^*)) \subset \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)})$$

coincides with the kernel of

$$\begin{pmatrix} [R(n, r)] \cdot (-) \\ F^* \end{pmatrix} : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

Furthermore

$$\text{Ker} \begin{pmatrix} \mathcal{I} \\ T_r^* \end{pmatrix} = \{(\delta_0 F^*(a), -\mathcal{I}_{r-1}(\delta_1 a)) \mid a \in \text{Rep}((\text{GL}_n)^{(1)})\}$$

Before we prove the Proposition and the Corollary, we need another tool, the *character map*

$$\begin{array}{ccc} \text{Rep}(\text{GL}_n) & \xrightarrow{\text{ch}} & \mathbb{Z}[X(T)] \\ [V] & \mapsto & \sum_{\lambda \in X(T)} \dim(V_\lambda) e(\lambda) \end{array}$$

where $e(\lambda)$ is the basis element of $\mathbb{Z}[X(T)]$ corresponding to $\lambda \in X(T)$. Due to the highest weight characterization of the $L(\lambda)$, the character map maps this \mathbb{Z} -basis of $\text{Rep}(\text{GL}_n)$ to a linearly independent set in $\mathbb{Z}[X(T)]$. Hence it is injective. It is well known that its image is precisely $\mathbb{Z}[X(T)]^W$ where $W = S_n$ is the Weyl group.

Now let us write the indeterminant t_i for $e(\varepsilon_i)$ and denote by s_i the i -th elementary symmetric polynomial in the t_i . Then

$$\mathbb{Z}[X(T)] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

and

$$\mathbb{Z}[X(T)]^W = \mathbb{Z}[s_1, \dots, s_n, s_n^{-1}]$$

As $F^* : \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$ acts on the T -weights by the p -th power, it corresponds to the p -th Adams operation ψ^p on $\mathbb{Z}[X(T)]$ which is

given by $\psi^p(t_i) = t_i^p$. Note that

$$\begin{aligned} \text{ch}([\Lambda^i U]) &= s_i \\ \text{ch}(\delta_r) &= \sum_{i=0}^n (-1)^{n-i} s_i = \prod_{i=1}^n (t_i - 1) =: \delta \\ \text{ch}([R(n, r)]) &= \prod_{i=1}^n \frac{t_i^{p^r} - 1}{t_i - 1} =: U_r \end{aligned}$$

Hence

$$\text{ch}((F^r)^*(\delta_r)) = (\psi^p)^r(\delta) = U_r \delta = \text{ch}([R(n, r)]\delta_0)$$

which implies that $(\delta_0, -(F^{r-1})^*(\delta_r))$ lies in the kernel of

$$\left(\begin{array}{c} [R(n, r)] \cdot (-) \\ F^* \end{array} \right) : \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) \rightarrow \text{Rep}(\text{GL}_n)$$

Before we prove the whole Proposition, we first deduce the Corollary.

Proof of 7.5. According to the Proposition, the kernel of $\left(\begin{array}{c} [R(n, r)] \cdot (-) \\ F^* \end{array} \right)$ is generated by $(\delta_0, -(F^{r-1})^*(\delta_r))$ as an $\text{Rep}((\text{GL}_n)^{(1)})$ -module. Hence the kernel elements are those of the form

$$(\delta_0 F^*(a), -(F^{r-1})^*(\delta_r)a)$$

for all $a \in \text{Rep}((\text{GL}_n)^{(1)})$. Furthermore, $(\psi^p)^{r-1}(\delta) = U_{r-1}\delta$. Hence

$$(F^{r-1})^*(\delta_r)a = [R(n, r-1)^{(1)}]\delta_1 a = \text{res } \mathcal{I}_{r-1}(\delta_1 a)$$

lies in the image of

$$\text{res} : \text{Rep}(G(n, r-1)^{(1)}) \rightarrow \text{Rep}((\text{GL}_n)^{(1)})$$

That is,

$$\text{Ker} \left(\begin{array}{c} [R(n, r)] \cdot (-) \\ F^* \end{array} \right) \subset \text{Im}(\text{id} \oplus \text{res})$$

Now the assertion follows from the commutativity of

$$\begin{array}{ccc} \text{Rep}(\text{GL}_n) \oplus \text{Rep}(G(n, r-1)^{(1)}) & \xrightarrow{\left(\begin{array}{c} \mathcal{I} \\ \mathcal{I}_r^* \end{array} \right)} & \text{Rep}(G(n, r)) \\ \text{id} \oplus \text{res} \downarrow & & \downarrow \text{res} \\ \text{Rep}(\text{GL}_n) \oplus \text{Rep}((\text{GL}_n)^{(1)}) & \xrightarrow{\left(\begin{array}{c} [R(n, r)] \cdot (-) \\ F^* \end{array} \right)} & \text{Rep}(\text{GL}_n) \end{array}$$

and the injectivity of res. \square

Now we give the proof of the Proposition.

Proof of 7.4. We will prove the Proposition in terms of $\mathbb{Z}[X(T)]$. In fact, we prove the following claim: The kernel of the map

$$\left(\begin{array}{c} U_r \cdot (-) \\ \psi^p \end{array} \right) : \mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] \rightarrow \mathbb{Z}[X(T)]$$

coincides with

$$\{(\delta\psi^p(a), -(\psi^p)^{r-1}(\delta)a) \mid a \in \mathbb{Z}[X(T)]\}$$

Then the Proposition follows by passing to S_n -invariants as $\mathbb{Z}[X(T)]$ is factorial.

We start with the case $r = 1$. As ψ^p and $U_1 \cdot (-)$ are injective, it suffices to show

$$\mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}] \cap U_1 \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = (U_1 \delta) \mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$$

For this, define the ideals

$$P_i := \left\langle \frac{t_i^p - 1}{t_i - 1} \right\rangle \subset \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

and

$$Q_i := \langle t_i^p - 1 \rangle \mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$$

Both P_i and Q_i are prime ideals of height 1 as they are generated by irreducible elements in factorial rings. We claim that the inclusion

$$Q_i \subset P_i \cap \mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$$

is an equality. As $\mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$ is factorial, it is integrally closed. Thus the “going-down” Theorem [Mat86, Theorem 9.4] implies that $P_i \cap \mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$ is also a prime ideal of height 1. Whence the equality. As

$$\langle U_1 \rangle = P_1 \cdots P_n = P_1 \cap \dots \cap P_n \subset \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

and

$$\langle U_1 \delta \rangle = Q_1 \cdots Q_n = Q_1 \cap \dots \cap Q_n \subset \mathbb{Z}[t_1^{\pm p}, \dots, t_n^{\pm p}]$$

the case $r = 1$ follows.

Now we consider the case $r \geq 2$. Then we can factor our map as follows since $U_1 \psi^p(U_{r-1}) = U_r$.

$$\begin{array}{ccc} \mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] & \xrightarrow{\begin{pmatrix} U_r \cdot (-) \\ \psi^p \end{pmatrix}} & \mathbb{Z}[X(T)] \\ \downarrow \begin{pmatrix} \psi^p(U_{r-1}) \cdot (-) \oplus \text{id} \end{pmatrix} & \nearrow \begin{pmatrix} U_1 \cdot (-) \\ \psi^p \end{pmatrix} & \\ \mathbb{Z}[X(T)] \oplus \mathbb{Z}[X(T)] & & \end{array}$$

By the case $r = 1$ the kernel of the map $U_1 \cdot (-) + \psi^p$ consists of the elements of the form

$$(\delta \psi^p(a), -\delta a)$$

for $a \in \mathbb{Z}[X(T)]$. As $\psi^p(U_{r-1}) \cdot (-) \oplus \text{id}$ is injective and no prime factor $t_i - 1$ of δ divides $\psi^p(U_{r-1})$, the images of the elements of the kernel of $\begin{pmatrix} U_r \cdot (-) \\ \psi^p \end{pmatrix}$ are those of the above type where U_{r-1} divides a . As $\delta U_{r-1} = (\psi^p)^{r-1}(\delta)$, the claim for $r \geq 2$ follows. \square

REFERENCES

- [Nak92] Daniel K. Nakano, *Projective modules over Lie algebras of Cartan type*, Mem. Amer. Math. Soc. **98** (1992), no. 470, vi+84. MR1108120 (92k:17013)
- [Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid. MR879273 (88h:13001)
- [Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR2015057 (2004h:20061)
- [Abr96] William P. Abrams, *Representations of group schemes of Cartan type*, Comm. Algebra **24** (1996), no. 1, 1–14, DOI 10.1080/00927879608825553. MR1370522 (96k:17032)

- [Abr97] ———, *Representations of the group scheme $\text{Aut}(W_n)$* , Comm. Algebra **25** (1997), no. 9, 2765–2774, DOI 10.1080/00927879708826021. MR1458728 (98e:14048)
- [Kat70] Nicholas M. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 175–232. MR0291177 (45 #271)
- [DG80] Michel Demazure and Peter Gabriel, *Introduction to algebraic geometry and algebraic groups*, North-Holland Mathematics Studies, vol. 39, North-Holland Publishing Co., Amsterdam, 1980. Translated from the French by J. Bell. MR563524 (82e:14001)